Microscopic dynamics of the nonlinear Fokker-Planck equation: A phenomenological model

Lisa Borland*
Centro Brasileiro de Pesquisas Fisicas, Rua Dr. Xavier Sigaud 150, 22290-180 Rio de Janeiro, Rio de Janeiro, Brazil
(Received 16 December 1997)

We derive a phenomenological model of the underlying microscopic Langevin equation of the nonlinear Fokker-Planck equation, which is used to describe anomalous correlated diffusion. The resulting distribution-dependent stochastic equation is then analyzed and properties such as long-time scaling and the Hurst exponent are calculated both analytically and from simulations. Results of this microscopic theory are compared with those of fractional Brownian motion. [S1063-651X(98)00206-2]

PACS number(s): 66.10.Cb, 05.20.−y, 05.60.+w, 05.40.+j

I. INTRODUCTION

Anomalous diffusion is exhibited in a variety of physical systems and is therefore the subject of much current research. It can be observed, for example, in general systems such as plasma flow [1], porous media [2], and surface growth [2], as well as in more specific situations such as cytltrimethylammonium bromide miscelles dissolved in salted water [3] and NMR relaxometry of liquids in porous glasses [4]. The main characteristic of anomalous diffusion is the fact that the mean squared displacement is not proportional to time t but rather to some power of t. If the scaling is faster than t, then we say that the system is superdiffusive; if it is slower than t, we say that it is subdiffusive. The underlying mechanisms giving rise to anomalous diffusion may differ depending on the physical system. For example, the Lévy-type superdiffusion, whose mean squared displacement is infinite but possesses a well-defined anomalous scaling, is different from the correlated anomalous diffusion that describes transport in a porous medium. While a diffusion equation with fractional derivatives may be used to describe the Lévy-type diffusion [5], a *nonlinear* Fokker-Planck diffusion equation has been proposed for those systems with correlated anomalous diffusion [6–9].

An interesting feature of the nonlinear Fokker-Planck equation is that its exact stationary solutions, and some particular time-dependent solutions, are just those distributions that maximize the generalized entropy recently proposed by Tsallis [10]. That nonextensive entropy (inspired by multifractals), together with its associated generalized thermostatistics, has also been used to provide a thermostatistical basis for Lévy-type anomalous diffusion [11]. It is of course highly desirable to have these different types of diffusion related to a common general theory. In fact, there is a growing body of evidence for the physical relevance of that generalized thermostatistics in a variety of fields. It has been used, for example, to successfully study turbulence [12], cosmology [13], self-gravitating systems [14], linear response theory [15], the solar neutrino problem [16], and bremsstrahlung [17] among many other interesting physical systems. It has also been shown to be intimately related to the scaling properties of multifractal attractors [18].

The explicit form of the nonlinear Fokker-Planck equation is given by

$$\frac{df^{\mu}}{dt} = -\frac{d}{dx}(Kf^{\mu}) + Q\frac{d^2}{dx^2}(f^{\nu}), \tag{1}$$

where K is the drift coefficient, Q is the diffusion constant, and μ and ν are real numbers. t corresponds to time, while x denotes a state variable of the system. We assume that x is rescaled to be dimensionless. Equation (1) reduces into the standard, *linear*, Fokker-Planck equation for $\mu=\nu=1$. Plastino and Plastino [6] recently found exact time-dependent solutions in the form of the Tsallis distribution for $\mu=1$ and the drift force K proportional to x. Tsallis and Bukman [7] found exact solutions for the more general case of arbitrary μ and ν with a drift force of the form $K=k_1+k_2x$. Stariolo [19] studied the long-time behavior for systems with $K=0$ and arbitrary μ and ν. Also, Compte *et al.* [8,9] have studied solutions of a similar form in several dimensions and under shear flows. Note also that it has been shown [20] that even the standard linear Fokker-Planck equation may give rise to the stationary Tsallis distribution in a variety of peculiar cases. However, the linear Fokker-Planck equation typically leads to normal diffusion and will not be discussed in the current context.

Up until now, most of the discussion of correlated anomalous diffusion has been done on the macroscopic level, based on diffusion equations such as Eq. (1). We now know a substantial amount of information about the properties of the probability distribution that satisfies the Fokker-Planck equation of that form. However, there has yet been little or no effort in defining and studying the underlying microscopic dynamics that ultimately gives rise to a macroscopic level of description. This is the main objective of the current work. We shall derive and analyze the underlying stochastic Langevin equation that corresponds to the nonlinear Fokker-Planck-like equation presented below. The derivation, which is done in Sec. II, is consistent with the standard theory of stochastic processes and Fokker-Planck equations. In Sec. III we discuss solutions and realizations of the microscopic stochastic dynamics and in Sec. IV we study the long-time scaling behavior of the system. We then compare these results with those obtained for the well-known microscopic system of fractional Brownian motion, which gives rise to anoma-

*Electronic address: lisa@cat.cbpf.br

 © 1998 The American Physical Society

lous diffusion with a scaling power proportional to the Hurst coefficient H of the process. Finally, a discussion of the work is presented in Sec. V.

II. THE LANGEVIN EQUATION

A. The $\mu=1$ case

First, let us start by deriving the nonlinear Fokker-Planck equation for $\mu=1$, namely,

$$\frac{df}{dt}=-\frac{d}{dx}(Kf)+Q\frac{d^2}{dx^2}(f^{\nu}), \tag{2}$$

from an arbitrary stochastic Ito-Langevin equation of the form

$$\frac{dx}{dt}=K(x,t)+g(x,t)\,\eta(t). \tag{3}$$

We as yet make no assumptions on the noise η other than that

$$\langle \eta\rangle=0. \tag{4}$$

Also, let K and g be arbitrary functions. We proceed along the same lines as for the standard linear case [21]. The following relation holds:

$$f(x,t+\Delta t)=\int P(x,t+\Delta t|x',t)f(x',t)dx', \tag{5}$$

where $f(x,t)$ is the probability distribution of the particle having value x at time t and P denotes the transition probability between state values. The idea now is to set $x=x'+\Delta x$ and to expand the integrand in Eq. (5) into a Taylor series for small Δx. One obtains

$$P(x,t+\Delta t|x',t)f(x',t)=\left(-\Delta x\frac{d}{dx}P(x+\Delta x,t+\Delta t|x,t)+\frac{(\Delta x)^2}{2}\frac{d^2}{dx^2}P(x+\Delta x,t+\Delta t|,x,t)\right)f(x,t)+\cdots, \tag{6}$$

which can be integrated with respect to $d\Delta x$ to give

$$f(x,t+\Delta t)=\left(-\frac{d}{dx}\langle\Delta x\rangle+\frac{1}{2}\frac{d}{dx^2}\langle(\Delta x)^2\rangle+\cdots\right)f(x,t), \tag{7}$$

with

$$\langle\Delta x\rangle=\int \Delta x\ P(x,+\Delta x,t+\Delta t|x,t)d\Delta x \tag{8}$$

and likewise for $\langle(\Delta x)^2\rangle$. In the limit of $\Delta t\to 0$, these are the first two so-called Kramers-Moyal expansion coefficients. The problem now lies in evaluating $\langle\Delta x\rangle$ and $\langle(\Delta x)^2\rangle$. To this end we use the Langevin equation (3) so that

$$\langle\Delta x\rangle=\left\langle\int_t^{t+\Delta t}\dot{x}\ dt'\right\rangle \tag{9}$$

$$=\left\langle\int_t^{t+\Delta t}K(x,t')dt'\right\rangle+\left\langle\int_t^{t+\Delta t}g(x,t')dW(t')\right\rangle, \tag{10}$$

where special attention must be given to the variable $dW(t)=\eta(t)dt$, which defines the stochastic integration. The integration rules are slightly different, depending on whether one uses the Ito or Stratonovich calculus. In the Ito calculus, one assumes that the value of x at time t is determined by happenings prior to the stochastic force of $\eta(t)$. This results in the statistical independence of x and η, but also introduces some rules of calculus different from the usual ones. In the Stratonovich calculus, the variable x is valued at time t, whereas the stochastic variable η is evaluated at time intermediate to t and $t+dt$. This does not allow for a statistical independence between x and η, but does allow for the usual rules of calculus. In our work we prefer to use the Ito calculus, but point out that the results of one calculus can be mapped onto the other quite easily.

Within the Ito calculus we can treat $g(x,t)$ and $dW(t)$ as statistically independent, so that after use of $\langle dW\rangle=0$ we obtain from Eq. (10)

$$\langle\Delta x\rangle=K(x,t)\Delta t. \tag{11}$$

Similarly, for $\langle(\Delta x)^2\rangle$ we get

$$\langle(\Delta x)^2\rangle=\int_t^{t+\Delta t}\int_t^{t+\Delta t}\langle g(x,t')g(x,t'')\rangle\langle dW(t')dW(t'')\rangle \tag{12}$$

where we have discarded terms of second order and higher in Δt. In the standard theory, one assumes that the noise is white noise such that

$$\langle dW(t')dW(t'')\rangle=\delta(t'-t'')dt'. \tag{13}$$

In that case it is easy to see that

$$\langle\Delta x^2\rangle=g^2(x,t)\Delta t \tag{14}$$

up to order Δt. Inserting the results of Eqs. (11) and (14) into Eq. (7) and taking the limit $\Delta t\to 0$ yields the standard Fokker-Planck equation

$$\frac{df}{dt}=-\frac{d}{dx}[K(x,t)f]+\frac{1}{2}\frac{d^2}{dx^2}[g^2(x,t)f], \tag{15}$$

with $g^2=Q$ in the simplest case of constant diffusion.

So far we have simply reviewed the standard derivation of the linear Fokker-Planck equation from a microscopic Ito-Langevin equation. Our quest now is to see if and how the derivation can be modified so that we instead obtain the *nonlinear* Fokker-Planck equation of Eq. (2). To achieve this we see that we must require all the results to stay the same, except that for $\langle(\Delta x)^2\rangle$ of Eq. (12) we must require

$$\int_t^{t+\Delta t}\int_t^{t+\Delta t}\langle g(x,t')g(x,t'')\rangle\langle dW(t')dW(t'')\rangle = Qf^{\nu-1}. \tag{16}$$

The question now is how this can be achieved. The most straightforward way that leaves the rest of the theory invariant is to assume that the noise satisfies the standard condition (13), which implies that

$$g(x,t)^2 = Qf^{\nu-1}(x,t). \tag{17}$$

This result appears at first quite counterintuitive. However, let us focus on the mathematics for a moment. It follows from our derivation that, for δ-correlated white noise, the only way in which the nonlinear Fokker-Planck equation can arise from a microscopic Langevin equation is if Eq. (17) holds. Furthermore, we see that an f dependence in g does not interfere with the derivation of the macroscopic differential equation for f, so this choice is possible within the existing theory. The resulting Ito-Langevin equation has the form

$$\frac{dx}{dt} = K(x,t) + \sqrt{Q}f(x,t)^{(\nu-1)/2}\eta(t), \tag{18}$$

where the evolution of f is given by the Fokker-Planck equation of equation (2). A trajectory of Eq. (18) is determined by both equations simultaneously. It is apparent that there is feedback from the macroscopic level of description of the system in terms of the probability distribution f to the microscopic kinetics.

B. Colored noise

Before discussing the physical interpretation of the result (18), let us return to Eq. (16) and see if there is any other way in which it could be satisfied. This would entail lifting the constraint that the noise $\eta(t)$ is δ-correlated Gaussian noise and studying what may happen then. To this end we quote well-known results [22]. If Gaussian distributed δ-correlated noise is used then the Fokker-Planck equation (15) is exact in the sense that the higher-order Kramers-Moyal coefficients of the Taylor expansion (7) are equal to zero. If one uses δ-correlated non-Gaussian noise then some of those terms typically do not vanish, but the linear Fokker-Planck equation is still a good approximation in most cases. No nonlinearities in f are introduced. Colored noise $h(t)$ with a finite correlation time proportional to γ^{-1} also results in a linear yet non-Markovian problem. The memory effects in the noise can typically be dealt with by introducing an additional variable into the system, such that

$$\frac{dx}{dt} = s(x,t) + g(x,t)h(t), \tag{19}$$

$$\frac{dh}{dt} = \tilde{s}(h) + \tilde{g}(h)\,\eta(t), \tag{20}$$

where η can now be treated as δ-correlated white noise and s, g, $\tilde{s}$, and $\tilde{g}$ are arbitrary functions. In other words, the introduction of colored noise only raises the dimension of the problem. It does not introduce nonlinear orders of f in the Fokker-Planck equation. Furthermore, the Fokker-Planck equation corresponding to Eqs. (19) and (20) is certainly not even of the linear form (15) due to its non-Markovian nature.

As a remark, we point out that it can be shown [22] that for large γ one can neglect Eq. (20) and replace $h(t)$ with noise of Stratonovich type. Those results can then be transformed into the Ito calculus [22] so that the corresponding Ito process is given by

$$\frac{dx}{dt} = \left(s(x,t) + \frac{1}{2}g(x,t)\frac{d}{dx}g(x,t) \right)dt + g(x,t)dW(t). \tag{21}$$

By choosing to use g as above in Eq. (17) and

$$s(x,t) = K(x,t) - \frac{\nu-1}{4}\frac{df}{dx}f^{\nu-2}, \tag{22}$$

we see that this system also gives rise to the nonlinear Fokker-Planck equation (2). These results are essentially what we would have obtained if we had started out by using the Stratonovich calculus in the first place, even with δ-correlated white noise. This is because the additional term in the drift coefficient (22) is a direct result of the correlation between the variable x and the noise. This effect is, however, already incorporated within the Stratonovich calculus.

Based on the above discussion, we conclude that the nonlinear terms in Eq. (16) do not arise as an effect of colored noise. It is important to realize that colored noise turns the problem into a non-Markovian one, so that the form of the resulting Fokker-Planck equation itself will be non-Markovian and *not* of the form of Eq. (2), which is obviously Markovian. Only in the limit of white noise is a Markov approximation valid and we can recover the form of the nonlinear Fokker-Planck equation (2) within a Stratonovich representation. Here again, however, a microscopic dependence on f is required.

C. Interpretation

In summary then, in order to obtain the nonlinear Fokker-Planck equation (2) we are led to accept that the stochastic force depends, as in Eq. (16), on the probability distribution f. In the Ito calculus, within which we prefer to work in this paper, this results in the f-dependent Langevin equation (18). What does this f dependence mean and how can it arise?

Let us illustrate what is going on by visualizing the stochastic trajectories described by the Langevin equation (18) as the motion of a particle in a potential well defined by $V = -\int K(x)dx$. In the absence of a stochastic force the particle would sit still at the minimum of the well. In the presence of a stochastic force the particle gets knocked around, so that it may in principle traverse the entire well. If $\nu = 1$, standard Brownian motion is recovered. In that case, the sizes of the random kicks are uniform and do not depend on where in the well the particle happens to be. If one waits long enough, the entire well will become traversed. However, in the $\nu \neq 1$ cases, we see that the size of the random kicks changes in space and time. In particular, it will change such that highly frequented regions of the well will tend to have larger (or smaller) kicks, depending on the value of ν.

This creates a bias in the ergodic behavior of the system. Some regions of the well will become traversed at a much higher rate than others, while others will become forbidden. Though in principle the entire well may be traversed if one waits long enough, the dependence on the powers of probabilities serves to cleave the phase space of the system. An effectively nonergodic space is created for the stochastic system to exist in. The phase space might even have the character of something similar to a multifractal.

It is a harder task to explain the physical origin of the f dependence in the microscopic dynamics. For systems in which we interpret f as a density this poses no philosophical problems. The microscopics may well depend on actual densities. However, if we wish to interpret f as a probability distribution, then perhaps the natural place to look is to the treatment of the bath variables. Remember that the stochastic term in the Langevin equation is the net result of the interaction of the system variable x with a bath, whose variables have been appropriately eliminated. The classic works of Ford *et al.* [23] and Zwanzig [24] treat this problem for the standard case of $\mu=1$. Yet their result is strongly dependent on *ad hoc* assumptions about the interaction Hamiltonian of the bath variables together with their initial distribution. It may be possible to generalize those assumptions to result in the f-dependent term that we obtain in the nonlinear Ito-Langevin equation. That work is currently in preparation but beyond the scope of this paper. For now, we simply propose that the f-dependent term is *a phenomenological description of the interaction of the particle with the bath*. We assume that the bath variables are δ correlated, yet the entire dynamics is only defined on a (possibly multifractal) subset of phase space, which is modeled by the $f^{(\nu-1)/2}$ term. Of course, in a real experimental situation many different types of interactions between the bath and the system could give rise to the same phenomenological f-dependent description. Indeed, the exact form of these interactions must be guided by the physics of the particular system under study.

D. The $\mu\neq 1$ case

We now look at the nonlinear Fokker-Planck equation of the form (1) with general values of the parameter μ. The derivation remains the same, except that we are now seeking a differential equation for f^{μ}. In particular, this implies that we must start out with the relation

$$f^{\mu}(x,t+\Delta t)=\int P(x,t+\Delta t|x',t')f^{\mu}(x',t)dx', \quad (23)$$

instead of Eq. (5). Using the same arguments as before, we obtain the kinetic equation

$$\dot{x}=K(x)+\sqrt{Q}f(x,t)^{(\nu-\mu)/2}\eta(t), \quad (24)$$

where f satisfies the nonlinear Fokker-Planck equation (1). The important observation to make at this point is that although Eq. (24), together with Eq. (1), describes a kinetic Ito-Langevin equation, *the probability distribution of the process is not given by* f. Instead, the probability distribution of the process is given by f^{μ}. This can easily be seen as follows. Using the Ito-Langevin equation (24) we write down a Fokker-Planck equation in the usual way, with Eq. (5) as the starting point. We denote the probability distribution by $\tilde{f}$ and obtain

$$\frac{d\tilde{f}}{dt}=-\frac{d}{dx}K\tilde{f}+\frac{d}{dx^2}(f^{\nu-\mu}\tilde{f}), \quad (25)$$

where f satisfies the nonlinear Fokker-Planck equation (1). However, we can set $\nu-\mu=\mu(\tilde{\nu}-1)$, with $\tilde{\nu}=\nu/\mu$, resulting in

$$\frac{d\tilde{f}}{dt}=-\frac{d}{dx}K\tilde{f}+\frac{d}{dx^2}[(f^{\mu})^{\tilde{\nu}-1}\tilde{f}]. \quad (26)$$

However, we know from Eq. (1) that the solution to this equation is given by $\tilde{f}=f^{\mu}$. We have thus shown that the probability distribution of the kinetic process corresponding to the nonlinear Fokker-Planck equation (1) with general μ is given by f^{μ} and *not* by f. Note that although one was previously aware that the nonlinear Fokker-Planck equation (1) could be mapped onto the $\mu=1$ case by introducing a new variable $\tilde{f}=f^{\mu}$ and using $\nu=\nu/\mu$ [7], it was not pointed out that the probability distribution of the process is given by $\tilde{f}=f^{\mu}$ and not by f. This is an important distinction because it can otherwise lead to mistakes. For example, in the paper by Stariolo [19] f is used as the probability distribution from which certain long-time properties of the system are calculated. In particular, he finds that the system both violates the fluctuation-dissipation relationship as well as exhibits properties of aging. However, if instead the probability distribution f^{μ} is used, the aging effects disappear. The violation of the fluctuation-dissipation relationship is nevertheless still valid.

III. SOLUTIONS

A. Stationary nonlinear Fokker-Planck equation

Exact solutions to the nonlinear Fokker-Planck equation in form of the time-dependent Tsallis distribution have been found by Plastino and Plastino for $\mu=1$ [6] and Tsallis and Bukman for $\mu\neq 1$ [7]. However, for the reasons discussed above, we focus our attention on the $\mu=1$ case in most of what follows. Our system of interest is therefore given by the Langevin equation (18) together with Eq. (2). We emphasize that the f occurring in the Langevin equation is the solution to the nonlinear Fokker-Planck equation and not just any arbitrary distribution. For linear drift, this probability distribution is a Tsallis distribution. The form that satisfies the stationary nonlinear Fokker-Planck equation is

$$f_q(x)=\frac{1}{Z_q}[1-\beta(1-q)V(x)]^{1/(1-q)}, \quad (27)$$

where Z_q takes care of normalization and $V(x)=\int K(x')dx'$ is the potential. The parameter β is given by

$$\beta=\frac{2}{Q}\frac{Z_q^{q-1}}{(2-q)} \quad (28)$$

and the relation

$$\nu = 2 - q \tag{29}$$

holds. Furthermore, this distribution maximizes the generalized Tsallis entropy [10], which has the form

$$S_q = \frac{1 - \int dx [f(x)]^q}{q - 1} \tag{30}$$

and is the foundation of the actively studied generalized thermostatistics. The Tsallis entropy reduces to the standard Boltzmann-Gibbs entropy $S = -\int f \ln f dx$ in the limit $q \to 1$ and possesses most of the same qualities, but not that of *extensivity*. The degree of this *nonadditivity* of the Tsallis entropy S_q is quantified by the Tsallis index q.

Because f is a positive quantity, we must impose $f_q(x) = 0$ if the term in square brackets on the right-hand side of Eq. (27) becomes negative. (This is known as the Tsallis cutoff.) By looking at the behavior of the stationary distribution $f_q(x)$ we see that this happens when

$$\frac{1 - q}{2 - q} > \frac{Q}{2 Z_q^{q-1} V(x)}, \tag{31}$$

where both Q and Z_q are under the physical constraints of positivity: Z_q because it is the integral of positive probabilities and Q because it is the square of the amplitudes of the stochastic fluctuations. For $q < 2$ this inequality can easily be satisfied because the left-hand side of the equation is bounded from above by 1. However, there is a singularity at $q = 2$, beyond which the left-hand side soars to infinity and becomes bounded from below by 1. We shall discuss the parameter region $q > 2$ later on. For now, let us focus on the $q < 2$ region.

B. Realizations for $q < 2$

The time-dependent solution to the nonlinear Fokker-Planck equation with linear drift has the form

$$f_q(x,t) = \frac{1}{Z_q(t)} \{1 - \beta(t)(1 - q)[x - x_M(t)]^2\}^{1/(1-q)}, \tag{32}$$

where

$$\frac{\beta(t)}{\beta(0)} = \left(\frac{Z_q(0)}{Z_q(t)} \right)^2. \tag{33}$$

Here we take $Z_q(0) = Z_q$ and $\beta(0) = \beta$ as defined above from the stationary distribution. $Z_q(t)$ and $x_M(t)$ are given as in Refs. [6],[7]. An example of this solution at two different times for different values of q is shown in Fig. 1. Correspondingly, simulations of the stochastic paths generated by the probability-dependent Ito-Langevin equation (18) can readily be computed. Several realizations are shown in Fig. 2 for different values of q but constant choice of K and Q. The main distinguishing feature between paths with different q values is, as expected, in the behavior of the fluctuations. Note that for $q = 1.5$ there are sharp swings in the fluctuations. The fluctuations are larger if the particle approaches a forbidden (or low probability) region of state space, which essentially serves to drive it back to the more favored (high-probability) region where the noise is lower and more confining. A quite different behavior is demonstrated by the $q = -4$ example. The amplitude of the noise is suppressed all

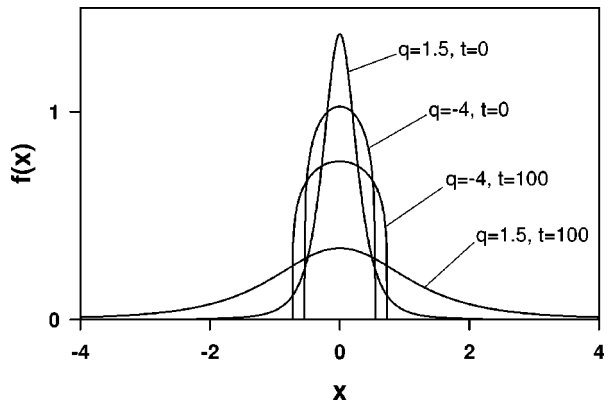

FIG. 1. Tsallis distributions, the exact solutions to the time-dependent nonlinear Fokker-Planck equation, are shown at times $t = 0$ and 100 for $q = 1.5$ and -4 with $K = -0.5x$ and $Q = 0.5$. x is a dimensionless state variable and t represents time in arbitrary units.

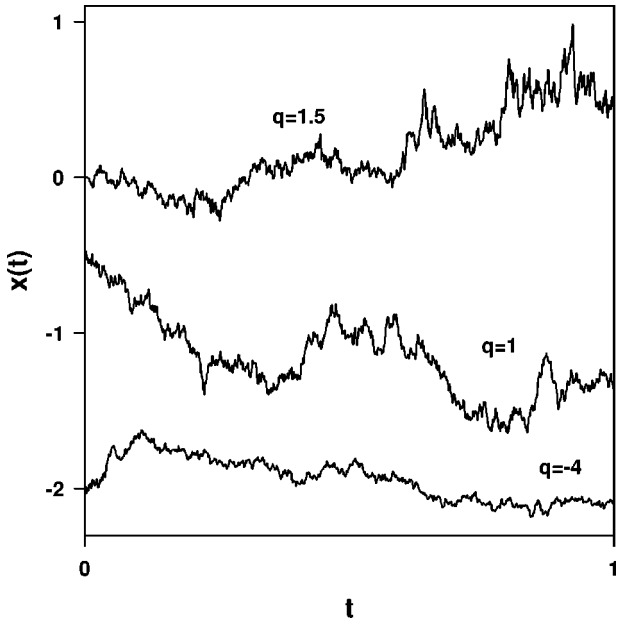

FIG. 2. Simulations of the stochastic trajectories obtained from the f-dependent Ito-Langevin equation (18) for different values of q with $K = -0.5x$ and $Q = 0.5$. The main distinguishing feature between paths is in the behavior of the fluctuations and the tendency of the paths to fill out the phase space. For $q = 1.5$ the fluctuations are larger if the particle approaches a forbidden (or low probability) region of state space, driving it back to the more favored (high-probability) region where the noise is lower and more confining. For $q = -4$, the amplitude of the noise is suppressed all over state space so that the particle stays close to the most probable (most frequented) region. For $q = 1$ we have normal Brownian motion of a particle with constant noise amplitude. The behavior of the trajectories is well reflected in the shapes of their corresponding probability distributions (see Fig. 1). x is a dimensionless state variable of the system and t corresponds to time in arbitrary units.

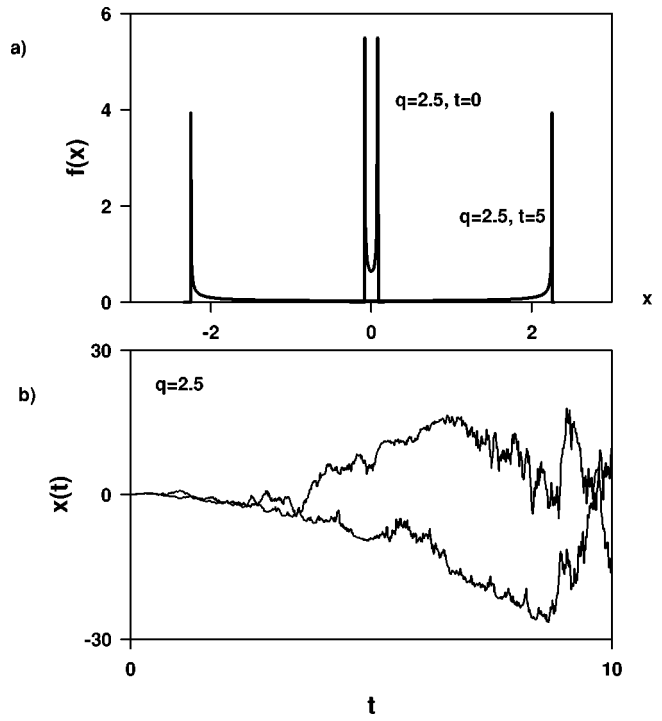

FIG. 3. Unphysical solutions for $q>2$. The distribution in (a) has a two-peaked shape, reflecting the symmetry breaking in the paths shown in (b). In time, the peaks in the distribution are limited to δ functions at $\pm\infty$. This narrowing of the allowed values of x is unphysical, defying the basic principle of entropy production. Furthermore, the stochastic paths soon get kicked into the forbidden $f=0$ region. x is a dimensionless state variable and t denotes time in arbitrary units.

over state space so that the particle stays close to the most probable (most frequented) region, with very low chance of departing. As a comparison, we also show the standard $q=1$ case, which represents the usual Brownian motion of a particle with constant noise amplitude. Note that for this system the noise appears quite homogeneous and the particle wanders off freely in any direction. The behavior of the trajectories is also well reflected in the shapes of their corresponding probability distributions, a few of which are shown in Fig. 1. Note the narrow shape of the Tsallis distribution for $q=-4$ as opposed to the broader one obtained for $q=1.5$. Furthermore, we point out that the long-time stationary probability distributions for the $q<2$ regime all have the appearance of stable, well-defined packets, even in the free-particle case ($K=0$).

C. Realizations for $q>2$

Now let us return to the case of $q>2$. As discussed in Ref. [7], the region $q>3$ leads to unphysical solutions due to the fact that it becomes impossible to normalize the temporally dependent Tsallis distribution. However, we must still look at the region $2<q<3$. For those values, the inequality (31) is satisfied only by a select choice of Q and V. The resulting distribution has the two-peaked shape presented in Fig. 3. Some sample trajectories are also presented. We see that initially there is a symmetry breaking. Because values $x=0$ have low probability, the paths diverge toward either $x>0$ or $x<0$. The behavior of the paths reflects the form of the distribution quite well. However, after a certain time, the peaks in the distribution become very narrow, limited to δ functions at $\pm\infty$. This narrowing down of the allowed values of x is unphysical because it implies that one would have more specific information about the location of the particle as time goes on, defying the basic principle of entropy production. This is reflected in the simulations of the trajectories in that the fluctuations in the stochastic paths quite quickly drive the system into the paradoxical region where $f=0$. This region is forbidden in the $q>2$ case because of the singularity that appears in the nonlinear Fokker-Planck equation for those values of q.

The physically more relevant form of the nonlinear Fokker-Planck equation in the regime $2<q<3$ is defined by

$$\dot{x}=-K(x)+\sqrt{Q}f(x,t)^{(\nu-1)/2}\eta(t), \tag{34}$$

$$\dot{f}=-\frac{d}{dx}(Kf)-Q\frac{d^2}{dx^2}(f^{\nu}), \tag{35}$$

with $\nu=2-q$. The negative sign of Q in the Fokker-Planck equation is motivated as follows. For $q>2$ the effective diffusion of the system is divergent. Therefore, our original postulate (5) must be changed to read

$$f(x,t+\Delta t)=-\int P(x,t+\Delta t|x',t)f(x',t)dx'. \tag{36}$$

This reflects the fact that the divergent diffusion coefficient has a repulsive effect. Interpreting f as a density, this is easier to understand. The density of particles at position x at time $t+\Delta t$ is not equal to the flux that flows in at that time, but rather equal to what is left after flux has been spewed out due to the repulsion. Using Eq. (36) as the starting point to derive a macroscopic Fokker-Planck equation for the Langevin equation in the same fashion as we did before in Sec. II, Eqs. (35) and (34) follow naturally.

The effectively negative diffusion coefficient $(-Q)$ in the nonlinear Fokker-Planck equation contributes a minus sign to the inequality of Eq. (31). Because of this, it is now easily fulfilled for a *general choice* of Q and V. A solution to Eq. (35) is shown in Fig. 4 for $K=2x$, $Q=0.1$, and $q=2.5$. Note that the Tsallis probability distribution disperses in a physical manner as time increases. Also, our simulations of the corresponding f-dependent stochastic paths appeared to be stable and well behaved.

IV. SCALING

A. The free particle

We shall use the kinetic Ito-Langevin equation (18) together with Eq. (2) to calculate the long-time scaling behavior of the process for $\mu=1$. To this end, we consider the free particle case where $K=0$ and Eq. (18) reduces to

$$\dot{x}=\sqrt{Q}f^{(\nu-1)/2}\eta(t). \tag{37}$$

The question is, if $\langle x(t)x(t')\rangle$ behaves in a certain way, then how does $\langle x(bt)x(bt')\rangle$ behave? We obtain

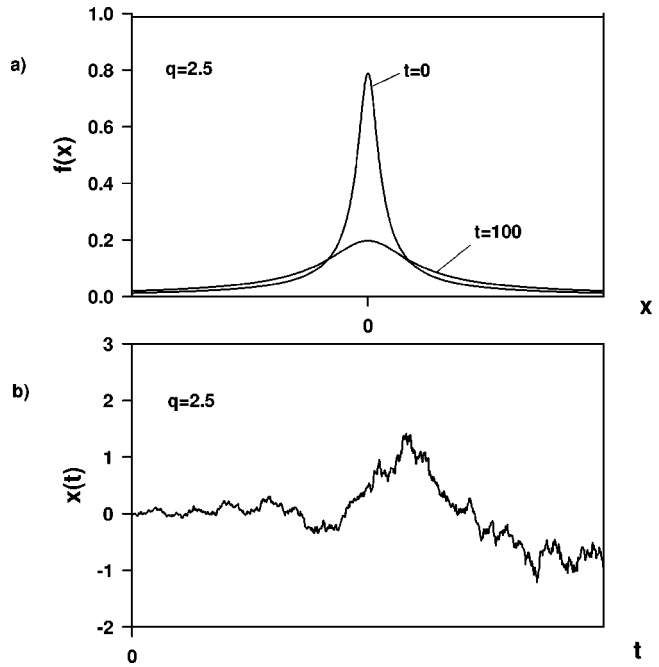

FIG. 4. Physical solutions for $q>2$ with $K=2x$, $Q=0.1$, and $q=2.5$. x is a dimensionless state variable and t corresponds to time in arbitrary units. The Tsallis probability distribution in (a) disperses in a physical manner as time increases. In (b), a well-behaved stochastic trajectory is shown.

$$x(t)=\sqrt{Q}\int_0^t f(x(\tau),\tau)^{(\nu-1)/2}dW(\tau) \tag{38}$$

and consequently

$$\begin{aligned}\langle x(t)x(t')\rangle &= Q\int_0^t\int_0^{t'}\langle f(x(\tau),\tau)^{(\nu-1)/2}f(x(\tau'),\tau')^{(\nu-1)/2}\rangle \\ &\quad\times\langle dW(\tau)dW(\tau')\rangle \\ &= Q\int_0^t f(x(\tau),\tau)^{\nu-1}d\tau=\langle x(t)^2\rangle, \qquad t=\min(t,t')\end{aligned} \tag{39}$$

where we have used the Ito calculus and the relation (13). We adopt the exact solution for $f(x,t)$ for $K=0$ from Refs. [7] and [19], which is of the form (32) and (33) with the normalization given by

$$Z_q(t)=\{Z_q(0)^{1+\nu}+2\nu(\nu+1)Q\beta(0)[Z_q(0)]^2t\}^{1/(1+\nu)}. \tag{40}$$

Furthermore, $x_M(t)=x_M(0)$ is the mean position and can be treated as a constant. Equation (39) becomes

$$\begin{aligned}\langle x(t)^2\rangle &= Q\int_0^\tau (a+\alpha\tau)^{(1-\nu)/(\nu+1)}-c[x(\tau)-x_M]^2 \\ &\quad\times(a+\alpha\tau)^{-1}d\tau,\end{aligned} \tag{41}$$

with $a=Z_q(0)^{1+\nu}$, $\alpha=2\nu(\nu+1)Q\beta(0)Z_q(0)^2$, and $c=\beta 0 Z_q(0)^2$. Using the relationship (29) between q and ν, we can alternatively express the exponent

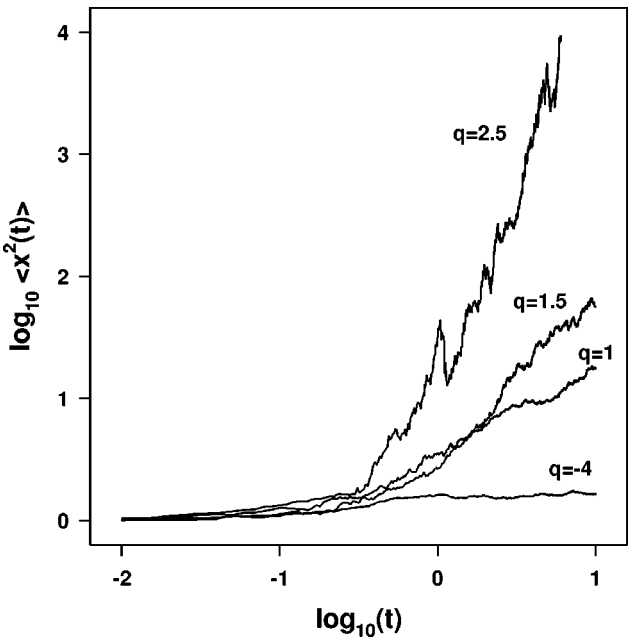

FIG. 5. Scaling coefficients from ensembles of simulated stochastic trajectories using Eq. (37) for different values of q were calculated. The results shown here in this $\log_{10}$-$\log_{10}$ plot indicate good agreement with the analytic result (44). For $q>1$ the scaling is superdiffusive and for $q<1$ it is subdiffusive.

$$\frac{1-\nu}{1+\nu}=\frac{q-1}{3-q}=F(q). \tag{42}$$

The integral in Eq. (41) is nontrivial to evaluate because of the term $x(\tau)$ occurring in the integrand. However, if $F(q)\geqslant -1$, which is satisfied for values $-\infty\leqslant q<3$ that are of physical interest to us here, then the integral will be dominated by the first term, which is easy to integrate. We obtain

$$\langle x(t)^2\rangle=Q\frac{3-q}{2\alpha}(a+\alpha t)^{2/(3-q)}. \tag{43}$$

Consequently, we see that the process satisfies the scaling relationship

$$\langle x(bt)^2\rangle=b^{2/(3-q)}\langle x(t)^2\rangle \tag{44}$$

for $t\to\infty$. This result reproduces that obtained by Stariolo [19] for $\mu=1$, which was calculated as an ensemble average using the solution to Eq. (1) with $\mu=1$. We also calculated the scaling coefficients from ensembles of simulated stochastic trajectories using Eq. (37) for different values of q. The log-log plot in Fig. 5 shows these results, which agree well with the analytic result of Eq. (44).

As a final remark, let us turn our attention to the $\mu\neq 1$ case of Eq. (24). We wish to calculate the scaling behavior for arbitrary μ. However, because Eq. (24), together with Eq. (1), can be mapped onto Eqs. (18) and (2) by substituting ν with $\tilde{\nu}=\nu/\mu$, we obtain the same results as in Eq. (44) for the $\mu=1$ case, except that we now have

$$q=2-\frac{\nu}{\mu} \tag{45}$$

instead of Eq. (29). This result does not agree with the $\mu \neq 1$ result obtained by Stariolo [19], which was calculated as an ensemble average based on using the function f as the probability distribution of the process described by the nonlinear Fokker-Planck equation (1). Our results indicate that the long-time scaling of $\langle x(t)x(t')\rangle$ does *not* exhibit the aging phenomena that he found. As mentioned earlier in this paper, we attribute this discrepancy to the fact that the true probability distribution of the process is given by $\tilde{f}=f^{\mu}$ and not by f.

B. Comparison with fractional Brownian motion

It has already been pointed out that both the nonlinear Fokker-Planck equation and a diffusion equation with fractional derivatives are good candidates for describing processes with anomalous diffusion [7,9,5]. The fractional diffusion equation is better suited to describe Levy-type processes, whereas the nonlinear diffusion equation treats correlated anomalous diffusion. Compte *et al.* [9] recently discussed a comparison of scaling within these two formalisms, based on the level of the diffusion equation. They showed that a process that scales as $\langle x(t)^2\rangle \propto t^{\gamma}$ may be described either as the result of nonlinear diffusion or by way of a diffusion equation with fractional derivatives. Their comparison was made based on studies of the macroscopic distribution equations, with the goal of elucidating which of the two formalisms provides a better description of anomalous diffusion under specific circumstances. Here we would like to make a similar comparison, but based on the analysis of the microscopic dynamics of the stochastic paths instead.

In the fractional derivative formalism, a stochastic path is described by [25]

$$x(t)=\Gamma \int_0^t (t-\tau)^{H-1/2} d\tau, \tag{46}$$

where H is the Hurst exponent defined in the interval $0<H<1$ and Γ is a positive constant. Normal Brownian motion is obtained with $H=\frac{1}{2}$. It is a well-known result that the process defined by Eq. (46) scales as

$$\langle x(bt)^2\rangle = b^{2H}\langle x(t)^2\rangle. \tag{47}$$

A comparison with our results of Eq. (44) shows that the process generated by the Langevin equation (37) scales in time as a fractional Brownian motion process where the Tsallis parameter q is related to the Hurst parameter H through

$$H=\frac{1}{3-q}. \tag{48}$$

However, this relationship is only valid for $-\infty<q<2$ because of the range of definition of the Hurst parameter. We can therefore say that if a process diffuses as $\langle x(t)^2\rangle \propto t^{\gamma}$ with $0<\gamma<2$ it may be described by either Eq. (46) or a process of the type (37). If $\gamma>2$ then it is more likely described by Eq. (37).

The relationship (48) between q and H is valid only when discussing the scaling behavior of the two processes. The stronger statement that the Hurst parameter of the f-dependent Langevin processes is equal to $(3-q)^{-1}$, which can easily be misread as Eq. (48), is *not* true. We calculated the Hurst coefficient for different values of q using simulated data obtained from numerical realizations of Eqs. (18) and (2). We used the original definition of Hurst (see Ref. [26]), namely, that

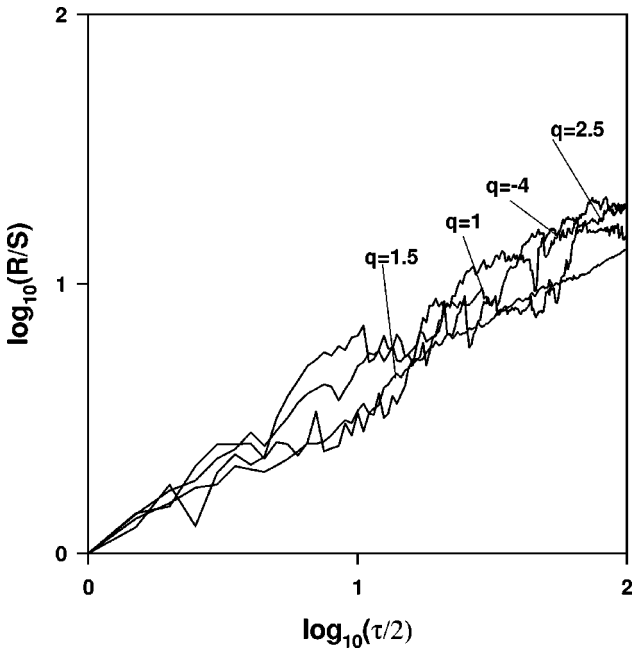

FIG. 6. Hurst coefficient H defined by $R/S=(\tau/2)^H$, calculated from simulated stochastic trajectories for different values of q. This $\log_{10}$-$\log_{10}$ plot shows that $H=0.5$ in all cases, reflecting the fact that there is no memory in these systems.

$$\frac{R}{S}=\left(\frac{\tau}{2}\right)^H, \tag{49}$$

where the range R is given by

$$R=\max_{1\leq t\leq \tau} X(t,\tau)-\min_{1\leq t\leq \tau} X(t,\tau), \tag{50}$$

with X equal to the accumulated departure from the mean of the stochastic increment within the time interval τ, namely,

$$X(t,\tau)=\sum_{i=1}^{t} [\zeta(i)-\langle\zeta\rangle_{\tau}], \tag{51}$$

with

$$\langle\zeta\rangle_{\tau}=\frac{1}{\tau}\sum_{t=1}^{\tau} \zeta(t). \tag{52}$$

Here $\zeta(t)$ represents the increment of the stochastic variable x of Eq. (37) in the interval δt. Similarly, the standard deviation S is defined as

$$S=\left(\frac{1}{\tau}\sum_{t=1}^{\tau} [\zeta(t)-\langle\zeta\rangle_{\tau}]^2\right)^{1/2}. \tag{53}$$

Using these definitions, we calculated R/S for different q. The results are shown in the log-log plot of Fig. 6. We see

that $H=0.5$ in all cases, independently of q. This result is to be expected because it reflects the fact that the f-dependent Langevin processes have no memory in time and are thus the result of completely uncorrelated statistical events. The fractional Brownian motion, on the other hand, has a memory, which is described by the t term under the integral in Eq. (46). These results indicate that a calculation of the Hurst coefficient for a process showing anomalous scaling of the form $\langle x(t)^2\rangle \propto t^{\gamma}$ may be used to discriminate whether it stems from a process within the framework of the nonlinear Fokker-Planck equation as opposed to a fractional Brownian motion process.

V. DISCUSSION

We have in this paper explored a form for the underlying microscopic Langevin equation that gives rise to the nonlinear Fokker-Planck equation. We have seen that the stochastic force in the Ito-Langevin equation depends on powers of the probability of the process itself. This dependence serves to cleave the phase space within which the process may traverse, essentially creating a nonergodic system. We suggest that an explanation for the dependence on the probability distribution of the system may be due to a particular and specific interaction between the system and the bath variables. Details of these ideas are, however, the subject of current work and lie outside the scope of this paper. If one instead interprets the probability distribution f as a distribution of real densities rather than a statistical description of the ensemble, there are no problems in understanding and interpreting our result. In such a case, the microscopic dynamics may well depend on a real field of densities in the form we propose.

Several realizations of the stochastic paths generated by the f-dependent Langevin equation in the case of linear drift force were simulated. In that case [6,7], the time-dependent solutions to the nonlinear Fokker-Planck equation are just those distributions that maximize the generalized entropy recently proposed by Tsallis. We analyzed the behavior of some f-dependent stochastic paths corresponding to different values of the Tsallis index q. This illustrates how the ergodic behavior of the system depends on the value of q. Our findings support other connections recently found between nonextensivity and ergodicity [27].

Furthermore, we studied the long-time scaling behavior of the free particle, based on the f-dependent microscopic dynamics of the system. No aging effects were found, contrary to the results of [19]. Our results were compared with those obtained for the well-known system of fractional Brownian motion, which gives rise to anomalous diffusion with a scaling power proportional to the Hurst coefficient H of the process. We found that although both systems lead to anomalous scaling behavior, the Hurst coefficient of the f-dependent Langevin process is always $H=0.5$ and does not depend on the scaling power (and therefore not on the Tsallis parameter q). This reflects the fact that our f-dependent Langevin equation does not contain long-term memory.

ACKNOWLEDGMENTS

I am indebted to C. Tsallis for inspiring discussions and helpful comments. I also thank A. K. Rajagopal for many fruitful conversations. This material is based upon work supported by the National Science Foundation under Grant No. INT-9703649.

[1] J. G. Berryman, J. Math. Phys. **18**, 2108 (1977).
[2] H. Spohn, J. Phys. I **3**, 69 (1993).
[3] A. Ott, J. P. Bouchaud, D. Langevin, and W. Urbach, Phys. Rev. Lett. **65**, 2201 (1990).
[4] O. V. Bychuk and B. O'Shaughnessy, Phys. Rev. Lett. **74**, 1795 (1995); S. Stapf *et al.*, *ibid.* **75**, 2855 (1995).
[5] A. Compte, Phys. Rev. E **53**, 4191 (1996).
[6] A. R. Plastino and A. Plastino, Physica A **222**, 347 (1995).
[7] C. Tsallis and D. J. Bukman, Phys. Rev. E **54**, R2197 (1996).
[8] A. Compte and D. Jou, J. Phys. A **29**, 4321 (1996).
[9] A. Compte, D. Jou, and Y. Katayama, J. Phys. A **30**, 1023 (1997).
[10] C. Tsallis, J. Stat. Phys. **52**, 479 (1988); E. M. F. Curado and C. Tsallis, J. Phys. A **24**, L69 (1991); **24**, 3187 (1991); **25**, 1019 (1992).
[11] P. A. Alemany and D. H. Zanette, Phys. Rev. E **49**, R956 (1994); D. H. Zanette and P. A. Alemany, Phys. Rev. Lett. **75**, 366 (1995); C. Tsallis, S. V. F. Levy, A. M. C. Souza, and R. Maynard, Phys. Rev. Lett. **75**, 3589 (1995); **77**, 5442(E) (1996).
[12] B. M. Boghosian, Phys. Rev. E **53**, 4754 (1996).
[13] V. H. Hamity and D. E. Barraco, Phys. Rev. Lett. **76**, 4664 (1996); D. F. Torres, H. Vucetich, and A. Plastino, *ibid.* **79**, 1588 (1997).
[14] A. R. Plastino and A. Plastino, Phys. Lett. A **174**, 384 (1993); **193**, 251 (1994).
[15] A. K. Rajagopal, Phys. Rev. Lett. **76**, 3469 (1996).
[16] G. Kaniadakis, A. Lavagno, and P. Quarati, Phys. Lett. B **369**, 308 (1996).
[17] C. Tsallis and A. M. C. de Souza, Phys. Lett. A **235**, 444 (1997).
[18] M. L. Lyra and C. Tsallis, Phys. Rev. Lett. **80**, 53 (1998).
[19] D. A. Stariolo, Phys. Rev. E **55**, 4806 (1997).
[20] L. Borland, Phys. Lett. A (to be published); D. A. Stariolo, Phys. Lett. A **185**, 262 (1994); G. Kaniadakis and P. Quarati, Physica A **237**, 229 (1997).
[21] H. Risken, *The Fokker-Planck Equation*, 2nd ed. (Springer, Berlin, 1984).
[22] C. W. Gardiner, *Handbook of Stochastic Methods*, 2nd ed. (Springer, Berlin, 1994).
[23] G. W. Ford, M. Kac, and P. Mazur, J. Math. Phys. **6**, 504 (1965).
[24] R. Zwanzig, J. Stat. Phys. **9**, 215 (1973).
[25] B. Mandelbrot and J. van Ness, SIAM (Soc. Ind. Appl. Math.) Rev. **10**, 422 (1968).
[26] J. Feder, *Fractals* (Plenum, New York, 1988).
[27] U. M. S. Costa, M. L. Lyra, A. R. Plastino, and C. Tsallis, Phys. Rev. E **56**, 245 (1997).